A diagrammatic solution to a problem on orders

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Abstract

In this work, we present a diagrammatic solution to the so called Richard Bird's Problem to find a first-order proof of the following:

Theorem (Richard Bird) For any two preorders X and Y, there exists a preorder Z such that, for any set S, the set of minimum elements w.r.t. Y of the set of minimum elements w.r.t X of the set S is the set of minimum elements w.r.t. Z of the set S (a is a minimum element of a set S w.r.t. a binary relation R when a is an element of S and is related by R to every element of the set S).

Our diagrammatic solution is a proof of this result using the Basic Graph Logic (BGL). Bird's theorem is an example of a result whose proof is "far more easily constructed when using the predicate calculus than in case of using the relational calculus", even though the relational calculus is "the symbolism par excellence to tackle such a theorem on relations" [W.H.J. Feijen, *One down for the relational calculus*, Technical Report WF148, 1991]. We present a proof of the same result in our diagrammatic calculus of relations as a *usability* test to BGL.

Let us introduce some notation, to state the problem we are about to prove in a more readable way. Given a relation $R \subseteq u \times u$ and a set $S \subseteq u$, define

$$\mathsf{min}_R S = \{x \in u : x \in S \text{ and } \forall y \in S, xRy\}.$$

A *preorder* is a reflexive and transitive binary relation. The problem, proposed by Richard Bird, is to use the predicate calculus to prove that:

Theorem (Richard Bird) For any two preorders X and Y, there exists a preorder Z such that, for any set S, $\mathsf{min}_Z S = \mathsf{min}_Y(\mathsf{min}_X S)$.

W.H.J. Feijen reports a proof of this theorem using the relacional calculus [3] and another proof using the predicate calculus [4]. Here, we present a diagrammatic proof of the same result, using the Basic Graph Logic (BGL) [1].

Fact 1 Let X and Y be binary relations. Then $Z = X \cap \left(\overline{X}^{-1} \cup Y\right)$ is reflexive when X and Y are reflexive.

We know that R is reflexive iff the identity relation I is included in R ($I \subseteq R$). Then we shall present a diagrammatic proof of $I \subseteq X, I \subseteq Y \implies I \subseteq Z$, given $Z = X \cap \left(\overline{X}^{-1} \cup Y\right)$. The diagrammatic versions of the hypotheses are:

$$\mp \;\subseteq\; - \xrightarrow{X} + \quad (X \text{ is reflexive}), \text{ and } \quad \mp \;\subseteq\; - \xrightarrow{Y} + \quad (Y \text{ are reflexive}).$$

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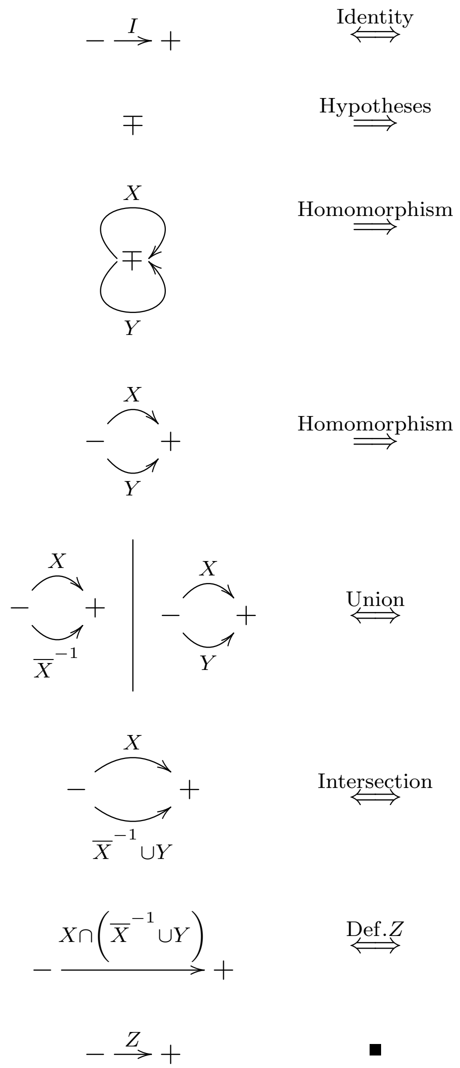
DIAGRAMMATIC PROOF (Fact 1):

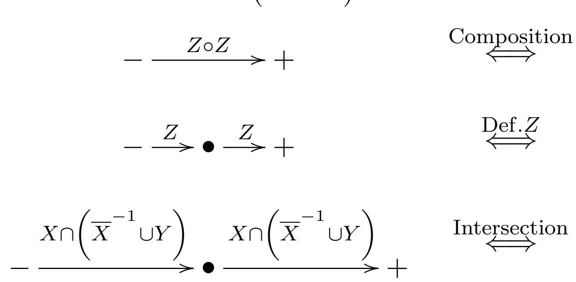


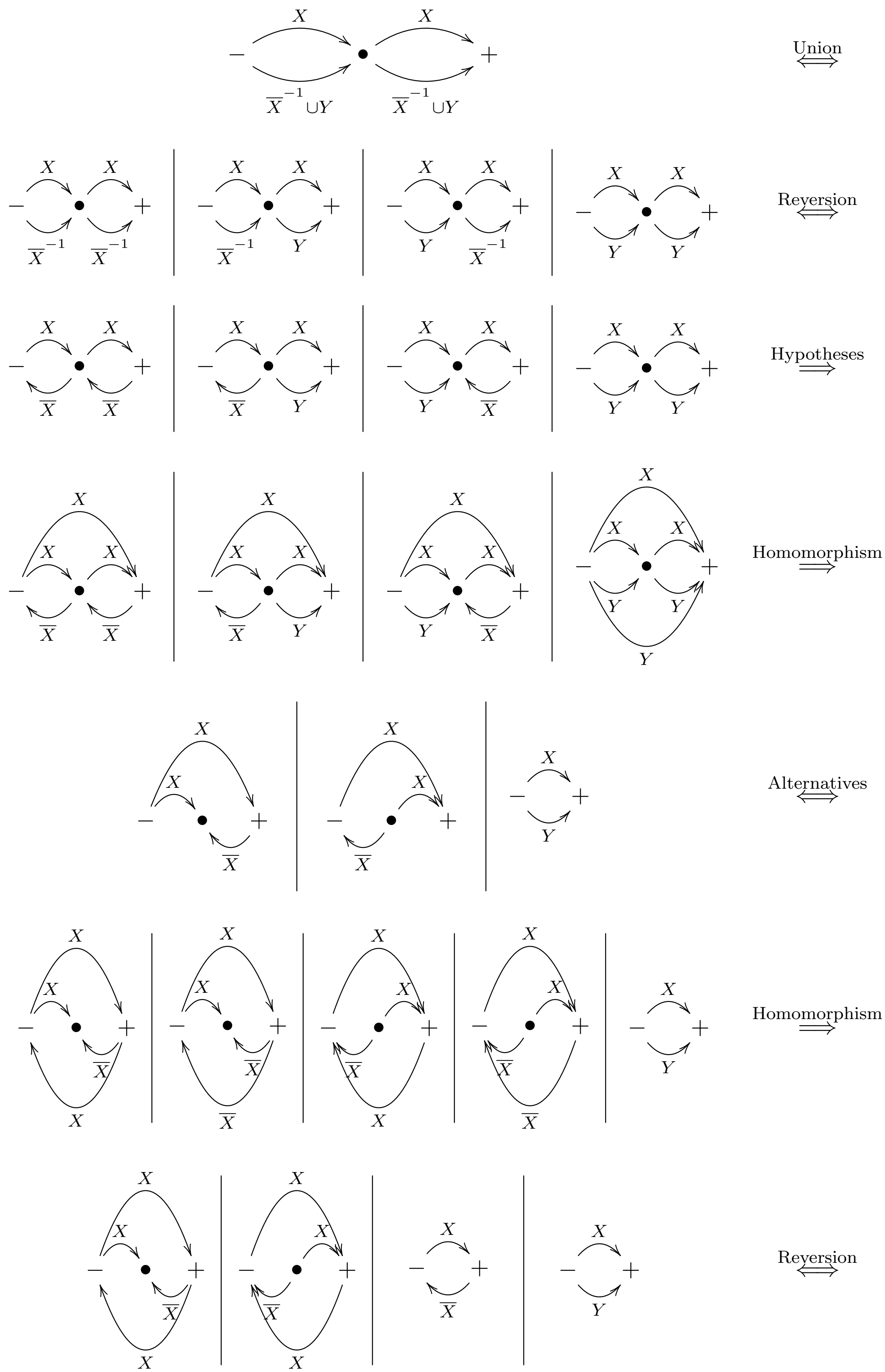
Fact 2 Let X and Y be binary relations. Then $Z = X \cap \left(\overline{X}^{-1} \cup Y\right)$ is transitive when X and Y are transitive.

We know that R is transitive iff the composition of R with R is included in R ($R \circ R \subseteq R$). Then we shall present a diagrammatic proof of $X \circ X \subseteq X, Y \circ Y \subseteq Y \implies Z \circ Z \subseteq Z$, given $Z = X \cap \left(\overline{X}^{-1} \cup Y\right)$. The diagrammatic versions of the hypotheses are:

$$- \xrightarrow{X} \bullet \xrightarrow{X} + \subseteq - \xrightarrow{X} + \quad (X \text{ is transitive}), \text{ and } \quad - \xrightarrow{Y} \bullet \xrightarrow{Y} + \subseteq - \xrightarrow{Y} + \quad (Y \text{ is transitive}).$$

DIAGRAMMATIC PROOF (Fact 2):




Union
Reversion
Hypotheses
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Alternatives
Homomorphism
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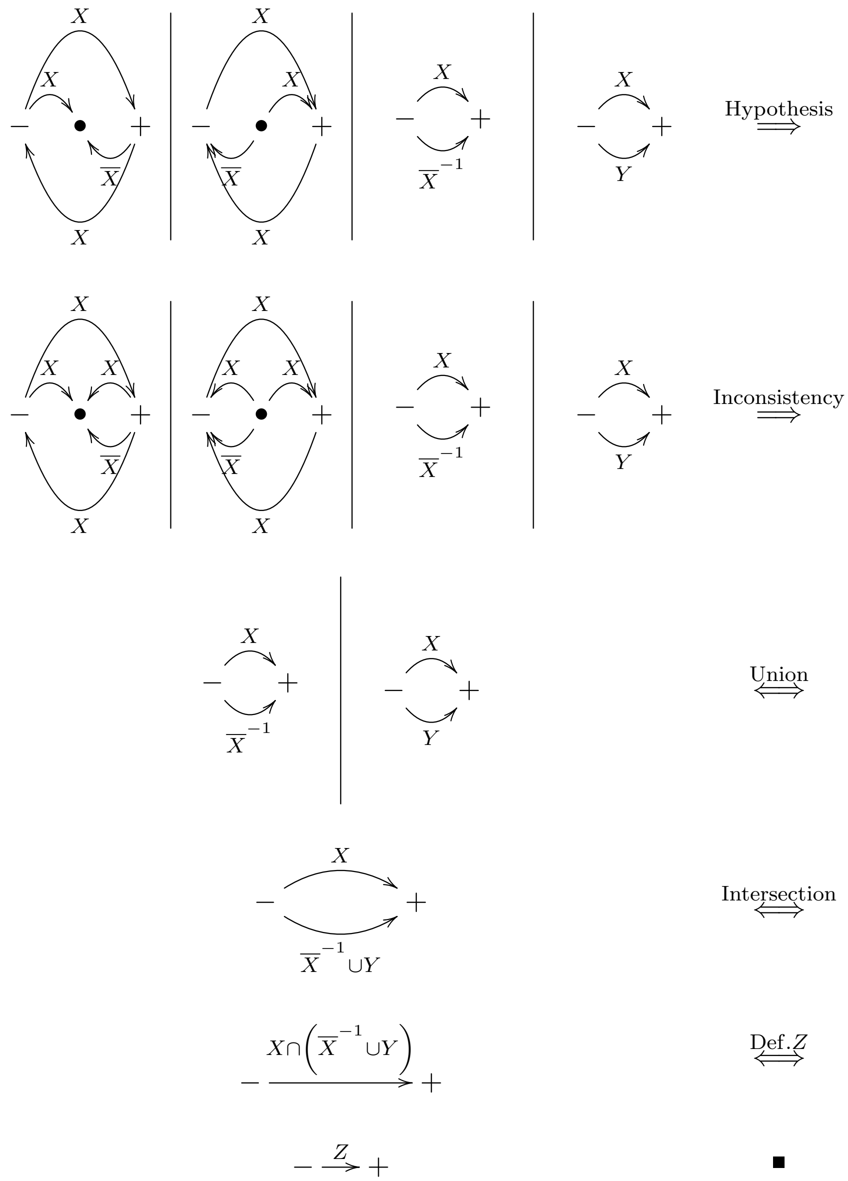


The proof of the next lemma uses almost only Boolean reasoning. We shall present a diagrammatic proof of it to illustrate both the diagrammatic manipulation of the complement and the following *heuristic* for producing a diagrammatic proof of an inclusion $L \subseteq R$:

1. Apply the elimination/introduction rules to obtain the diagrammatic normal form D_L of the left hand side L.

2. Apply the elimination/introduction rules to obtain the diagrammatic normal form D_R of the right hand side R.

3. Compare D_L and D_R, verifying if D_R covers D_L.

4. Stop if D_R covers D_L.

5. If D_R does not cover D_L, then

   (a) either apply a hypothesis to D_L and return to step 3,

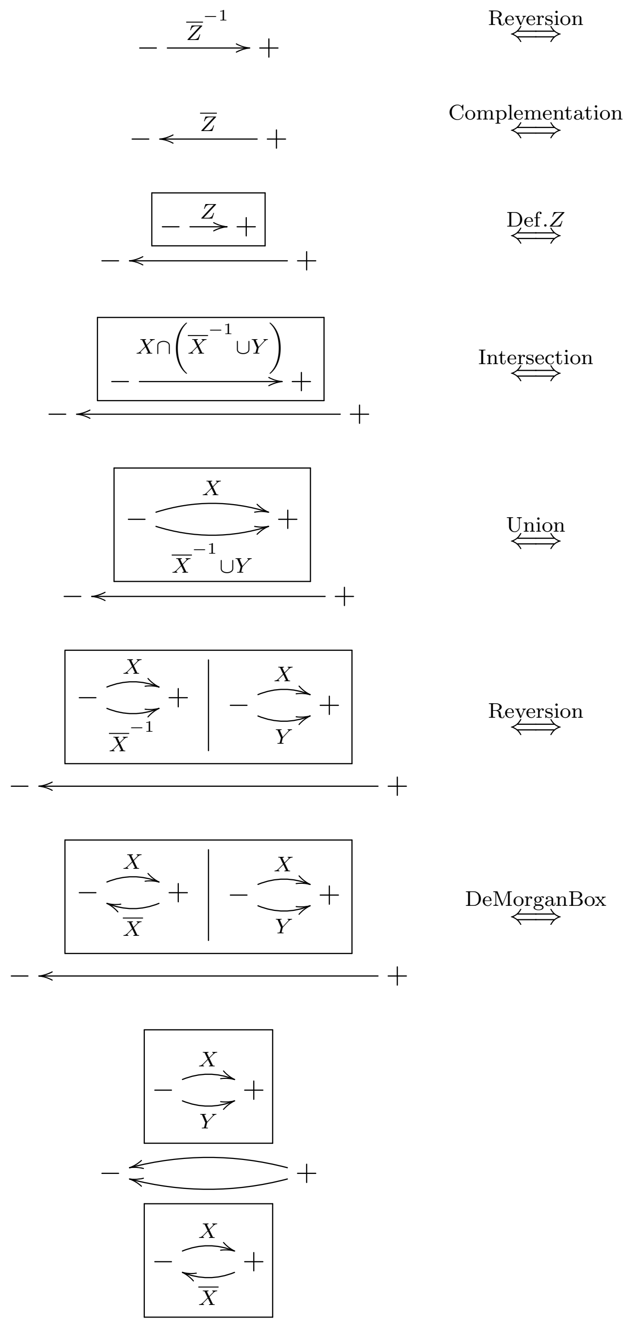
   (b) or apply the rule for Alternatives, or the rule for Inconsistency, and return to step 3.

The rule for Alternatives is used to introduce the desired parts of the goal diagram and the rule for Inconsistency is used to erase the non-desired ones.

Lemma Let X and Y be binary relations and $Z = X \cap \left(\overline{X}^{-1} \cup Y\right)$. Then $\overline{Z}^{-1} = \overline{X}^{-1} \cup \left(X \cap \overline{Y}^{-1}\right)$.

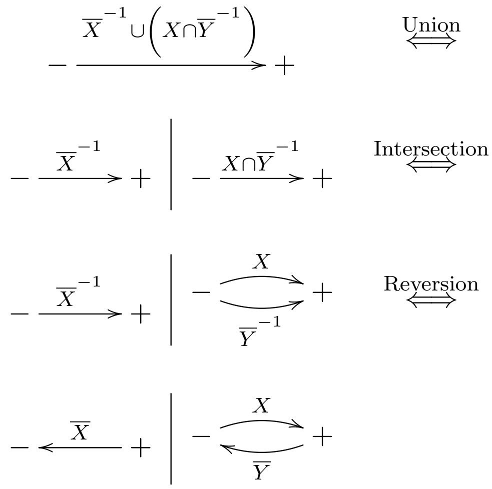
Diagrammatic proof:

1. Obtainint the normal form of $D_{\overline{Z}^{-1}}$:



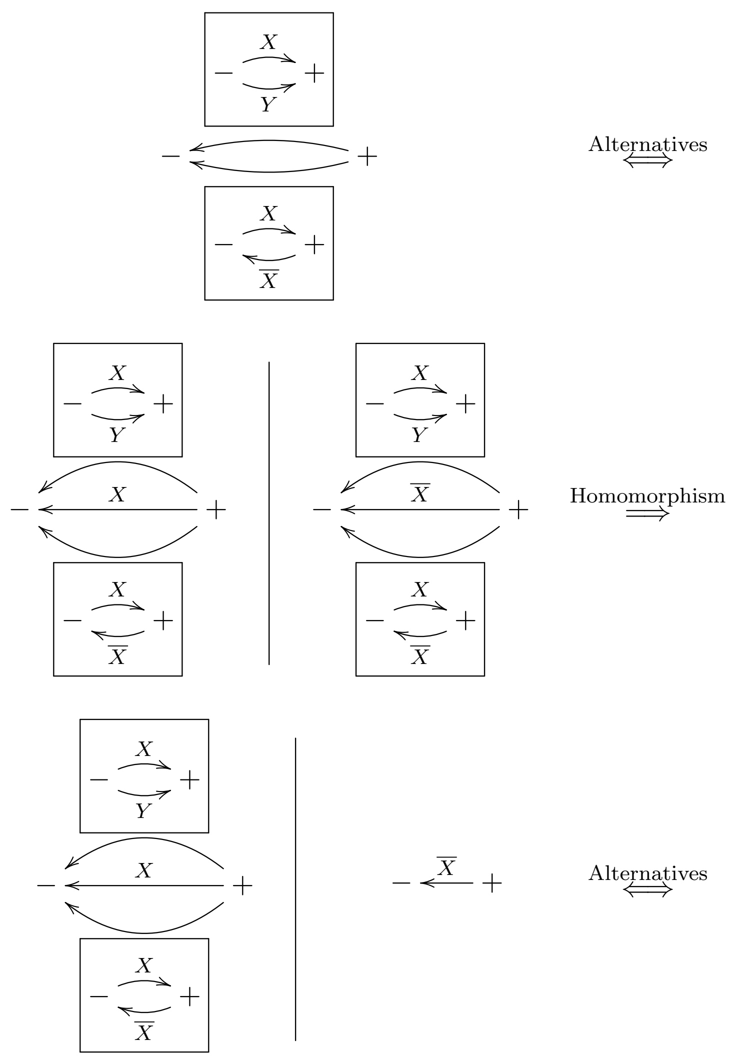
$\mathrm{NF}D_{\overline{Z}^{-1}}$

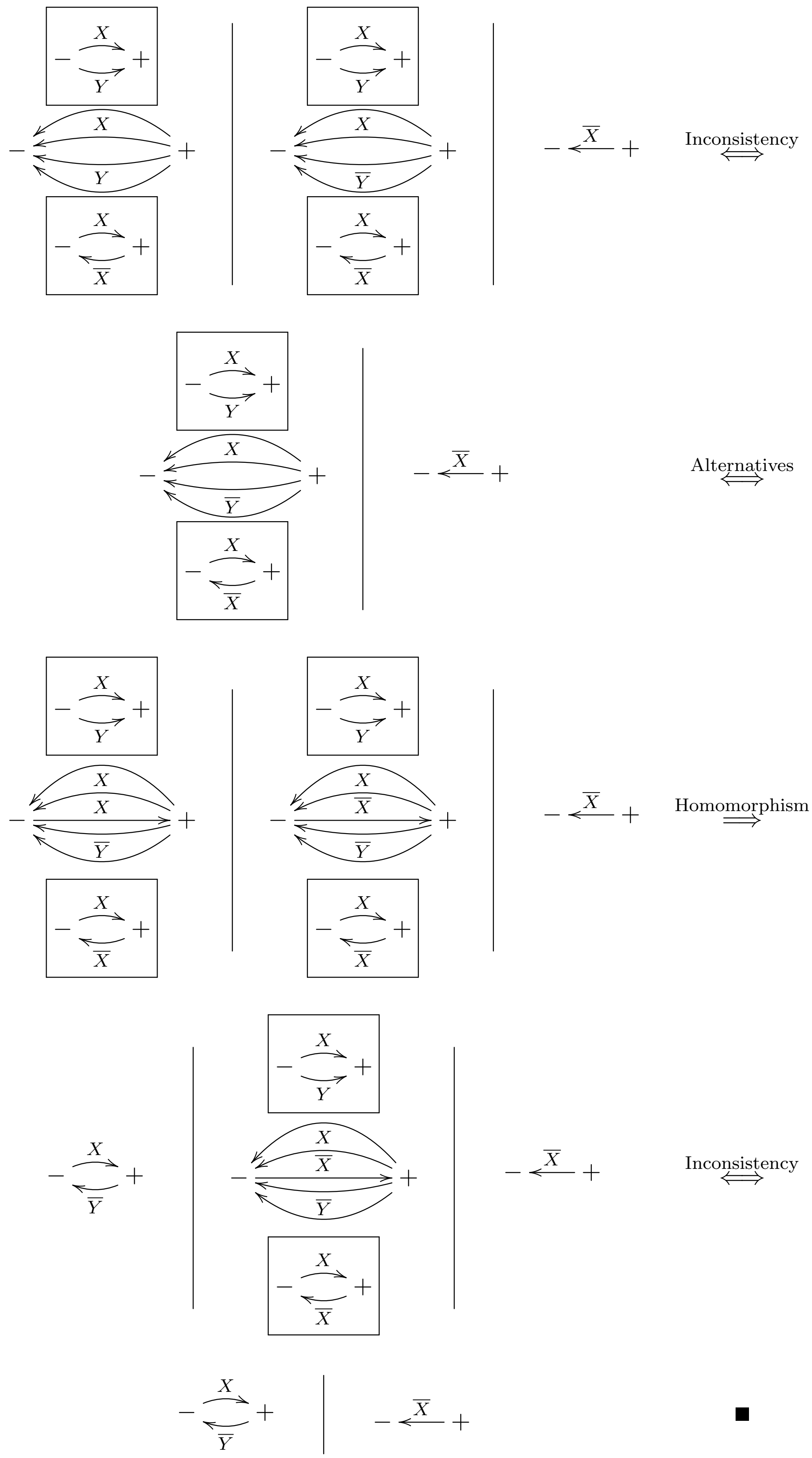
2. Obtainint the normal form of $D_{\overline{X}^{-1} \cup \left(X \cap \overline{Y}^{-1}\right)}$:



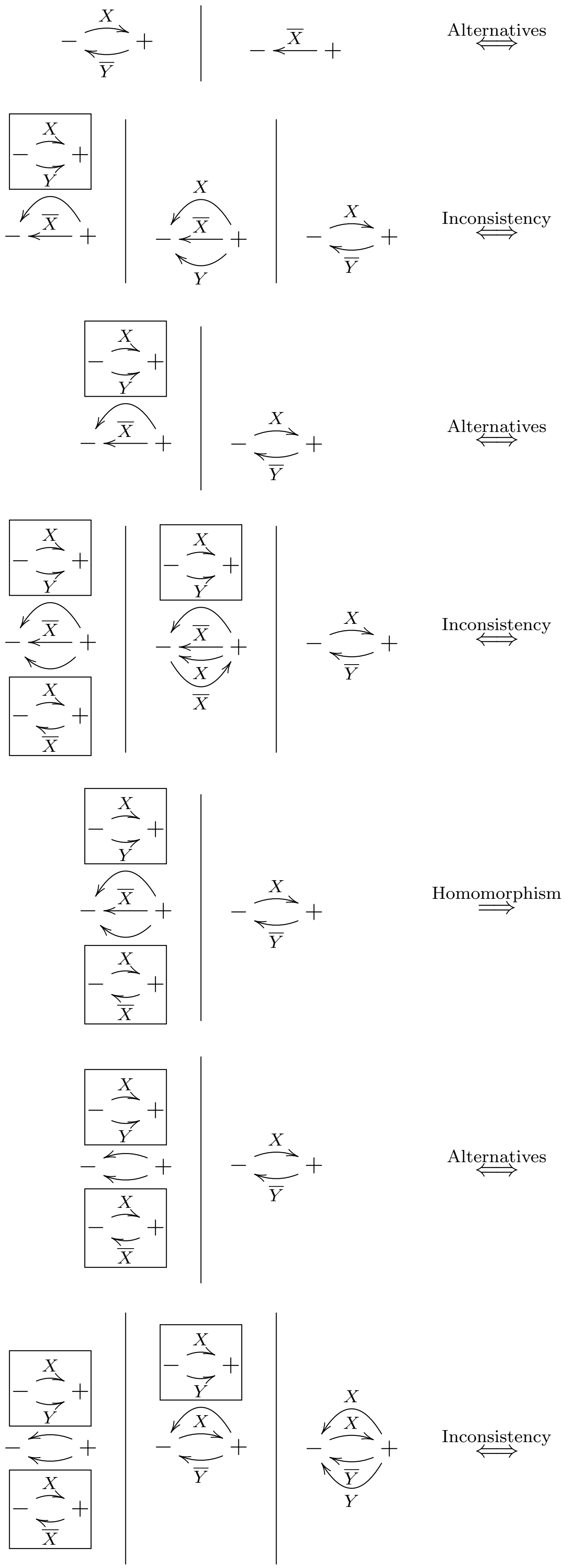
$$\mathrm{NF}D_{\overline{X}^{-1}\cup\left(X\cap\overline{Y}^{-1}\right)}$$

3-5. Diagrammatic proof of $\mathrm{NF}D_{\overline{Z}^{-1}} \subseteq \mathrm{NF}D_{\overline{X}^{-1}\cup\left(X\cap\overline{Y}^{-1}\right)}$:





3-5. Diagrammatic proof of $\mathrm{NF}D_{\overline{X}^{-1}\cup\left(X\cap\overline{Y}^{-1}\right)} \subseteq \mathrm{NF}D_{\overline{Z}^{-1}}$:


Alternatives
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Alternatives
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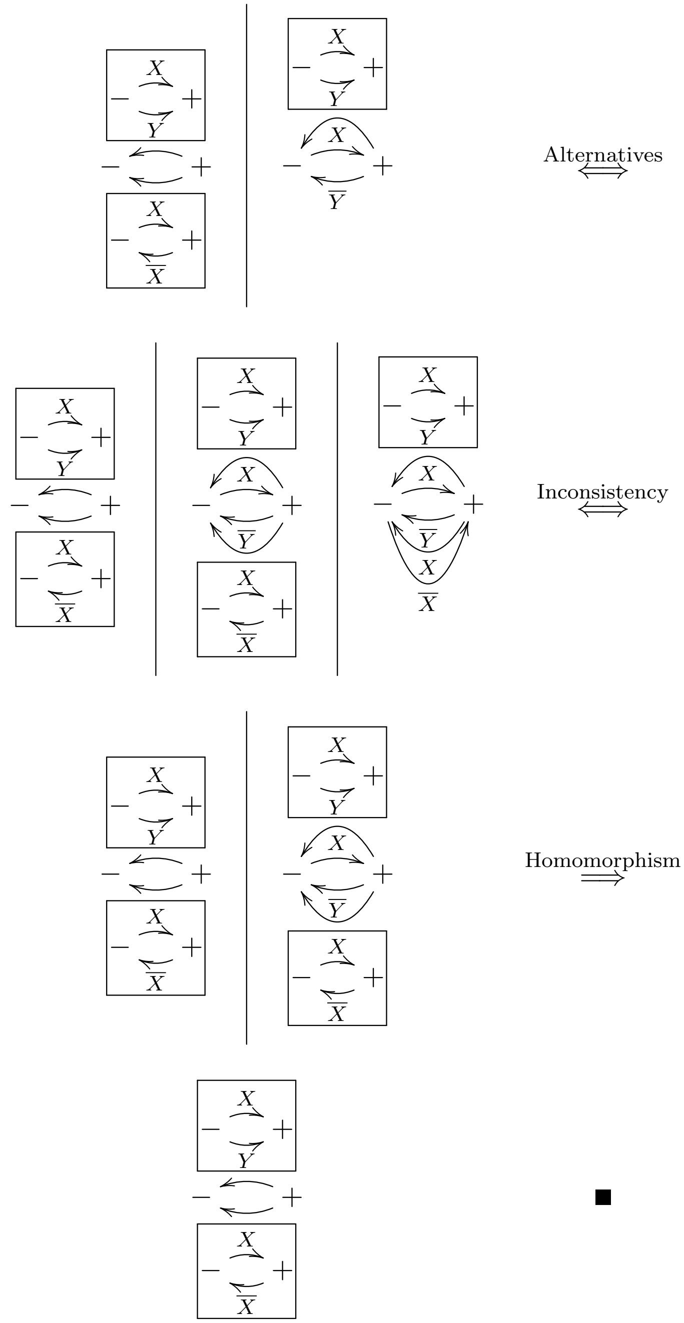
Inconsistency



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The last step to finish the proof of R. Bird's Theorem is to prove the following:

Fact 3 Let X and Y be binary relations and $Z = X \cap \left(\overline{X}^{-1} \cup Y\right)$. Then for all $S \subseteq u$:

$$\mathsf{min}_Z S = \mathsf{min}_Y(\mathsf{min}_X S).$$

Let us first define min as a binary relation. Given a relation $R \subseteq u \times u$, define

$$\mathsf{min}R = \{(S, x) : x \in S \text{ and } \forall y \in S, xRy\}.$$

Then we have that, for all $S \subseteq u$, for all $x \in u$:

$$(S, x) \in \mathsf{min}X \text{ iff } x \in \mathsf{min}_X S.$$

Using this, we conclude that, for all $S \subseteq u$, for all $x \in u$:

$$(S, x) \in \mathsf{min}X \cap \overline{\mathsf{min}X \circ \overline{Y}^{-1}} \text{ iff } x \in \mathsf{min}_Y(\mathsf{min}_X S).$$

Then Fact 3 can be reformulated as follows:

Fact 3' Let X and Y be binary relations and $Z = X \cap \left(\overline{X}^{-1} \cup Y\right)$. Then

$$\mathsf{min}Z = \mathsf{min}X \cap \overline{\mathsf{min}X \circ \overline{Y}^{-1}}.$$

We will now use the definition of min in terms of $\in$ proposed by Sharon Curtis [2]: given a relation $R \subseteq u \times u$, define

$$\mathsf{min}R \ = \ \in^{-1} \cap \overline{\in^{-1} \circ \overline{R}^{-1}}.$$

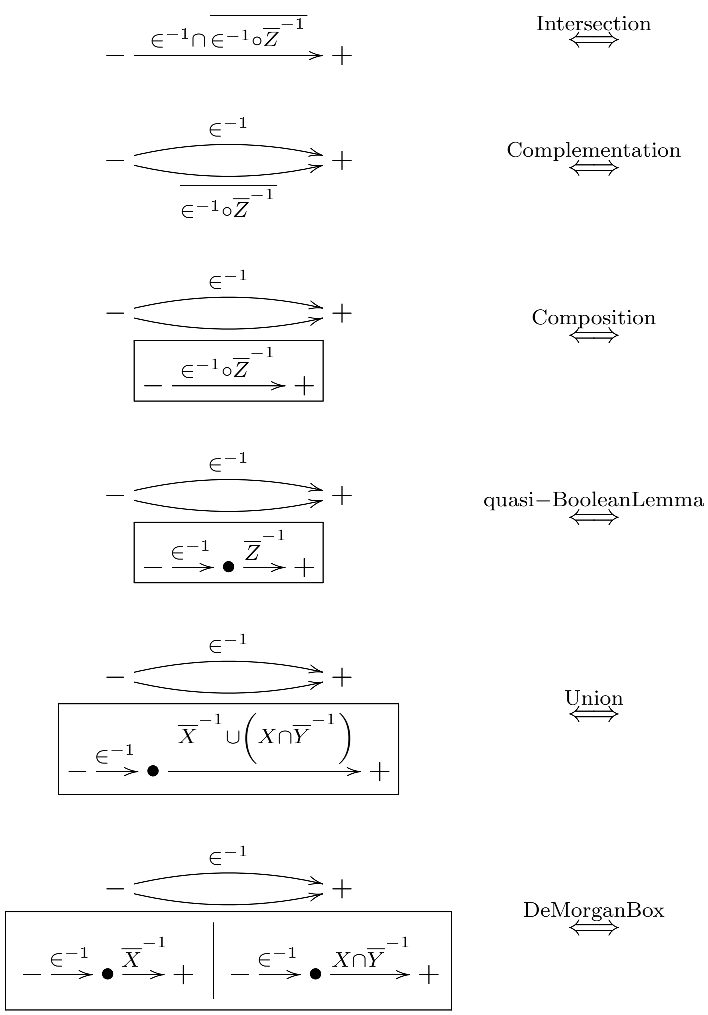
Using this definition, we reformulate Fact 3 once again:

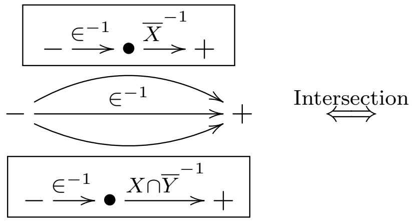
Fact 3" Let X and Y be binary relations and $Z = X \cap \left(\overline{X}^{-1} \cup Y\right)$. Then

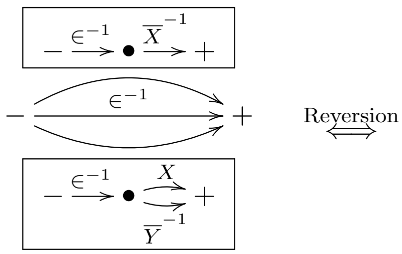
$$\in^{-1} \cap \overline{\in^{-1} \circ \overline{Z}^{-1}} \ = \ \in^{-1} \cap \overline{\in^{-1} \circ \overline{X}^{-1}} \cap \overline{\left(\in^{-1} \cap \overline{\in^{-1} \circ \overline{X}^{-1}}\right) \circ \overline{Y}^{-1}}.$$

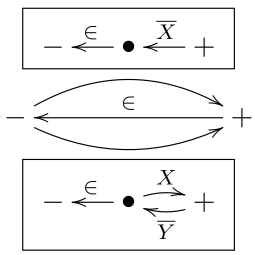
Diagrammatic proof of Fact 3":

1. Obtainint the normal form of $D_{\in^{-1} \cap \overline{\in^{-1} \circ \overline{Z}^{-1}}}$:



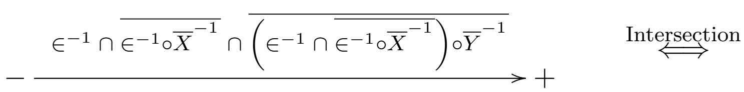


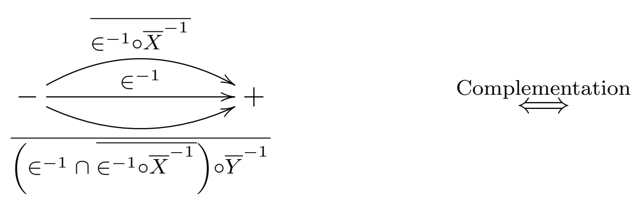


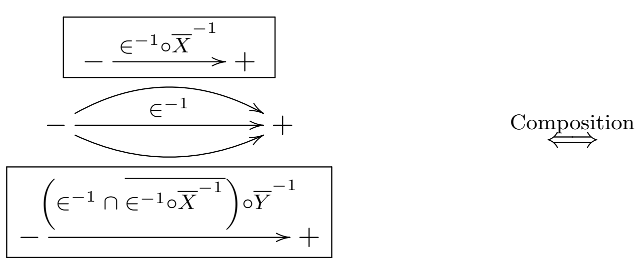


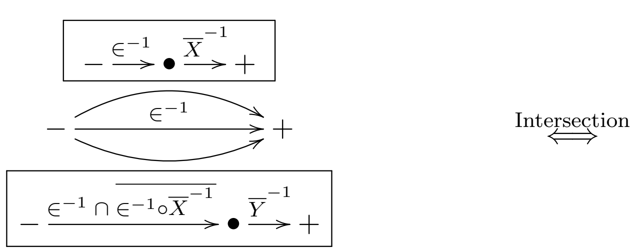
NF$D_{\in^{-1}\cap\overline{\in^{-1}\circ\overline{Z}^{-1}}}$

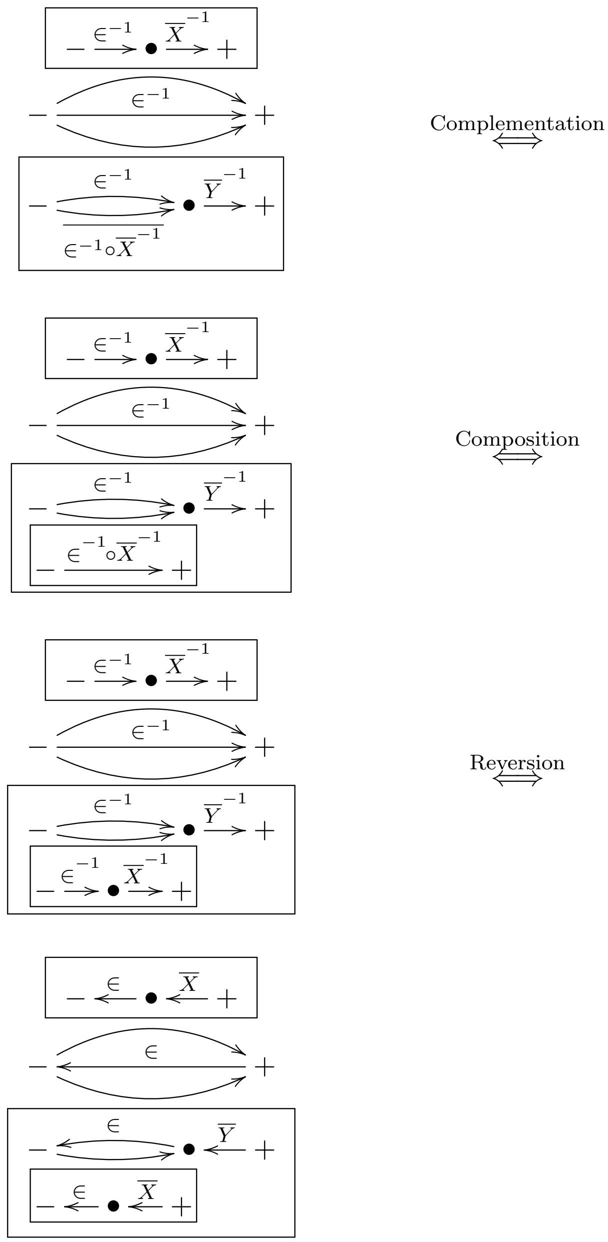
2. Obtainint the normal form of $D_{\in^{-1}\cap\overline{\in^{-1}\circ\overline{X}^{-1}}\cap\overline{\left(\in^{-1}\cap\overline{\in^{-1}\circ\overline{X}^{-1}}\right)\circ\overline{Y}^{-1}}}$:





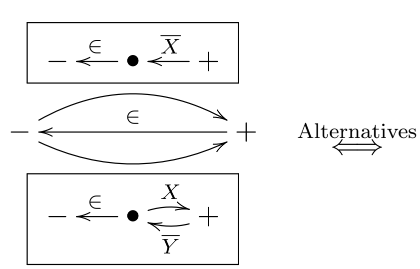


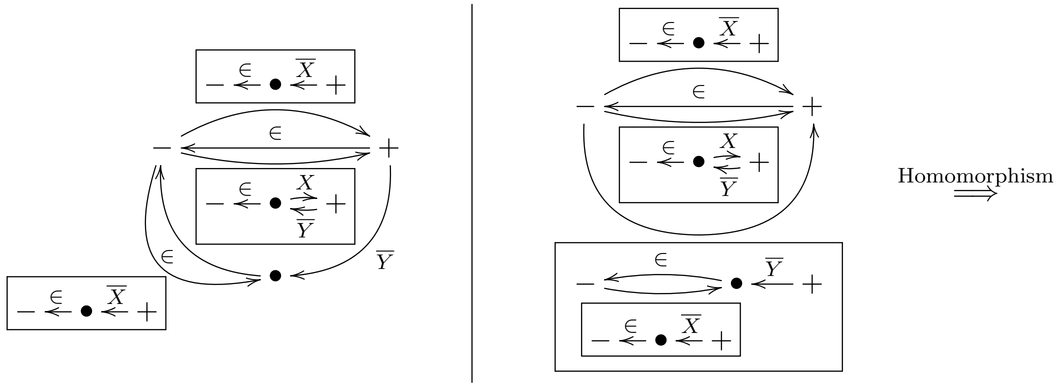


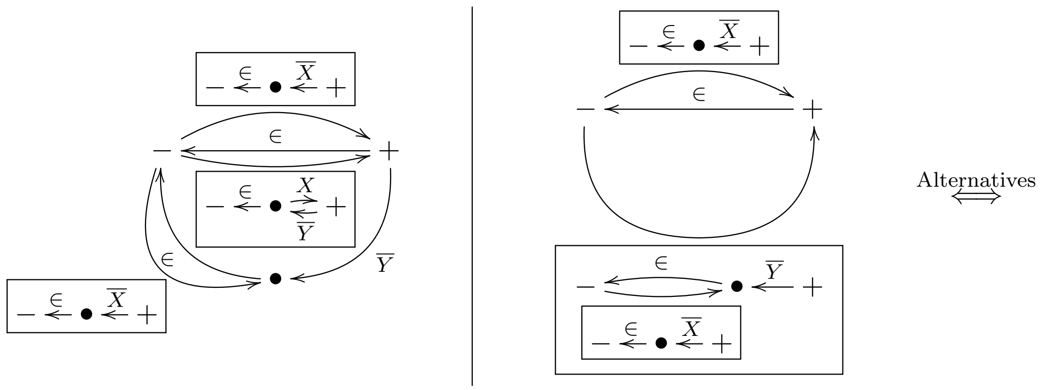


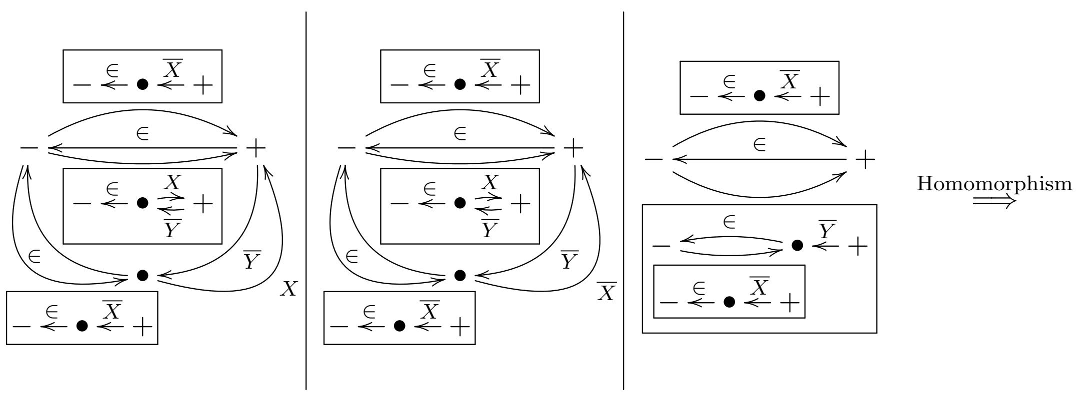
$$\mathrm{NF}D_{\in^{-1}\cap\overline{\in^{-1}\circ\overline{X}^{-1}}\cap\overline{\left(\in^{-1}\cap\overline{\in^{-1}\circ\overline{X}^{-1}}\right)\circ\overline{Y}^{-1}}}$$

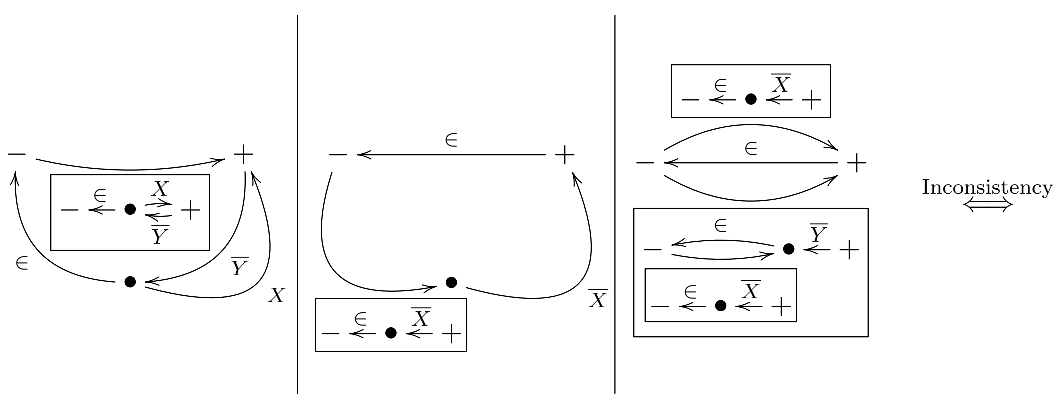
3-5. Diagrammatic proof of $\mathrm{NF}D_{\in^{-1}\cap\overline{\in^{-1}\circ\overline{Z}^{-1}}} \subseteq \mathrm{NF}D_{\in^{-1}\cap\overline{\in^{-1}\circ\overline{X}^{-1}}\cap\overline{\left(\in^{-1}\cap\overline{\in^{-1}\circ\overline{X}^{-1}}\right)\circ\overline{Y}^{-1}}}$:

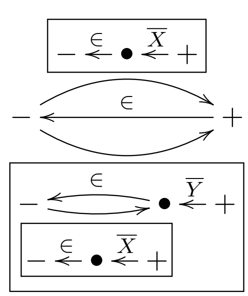



Homomorphism

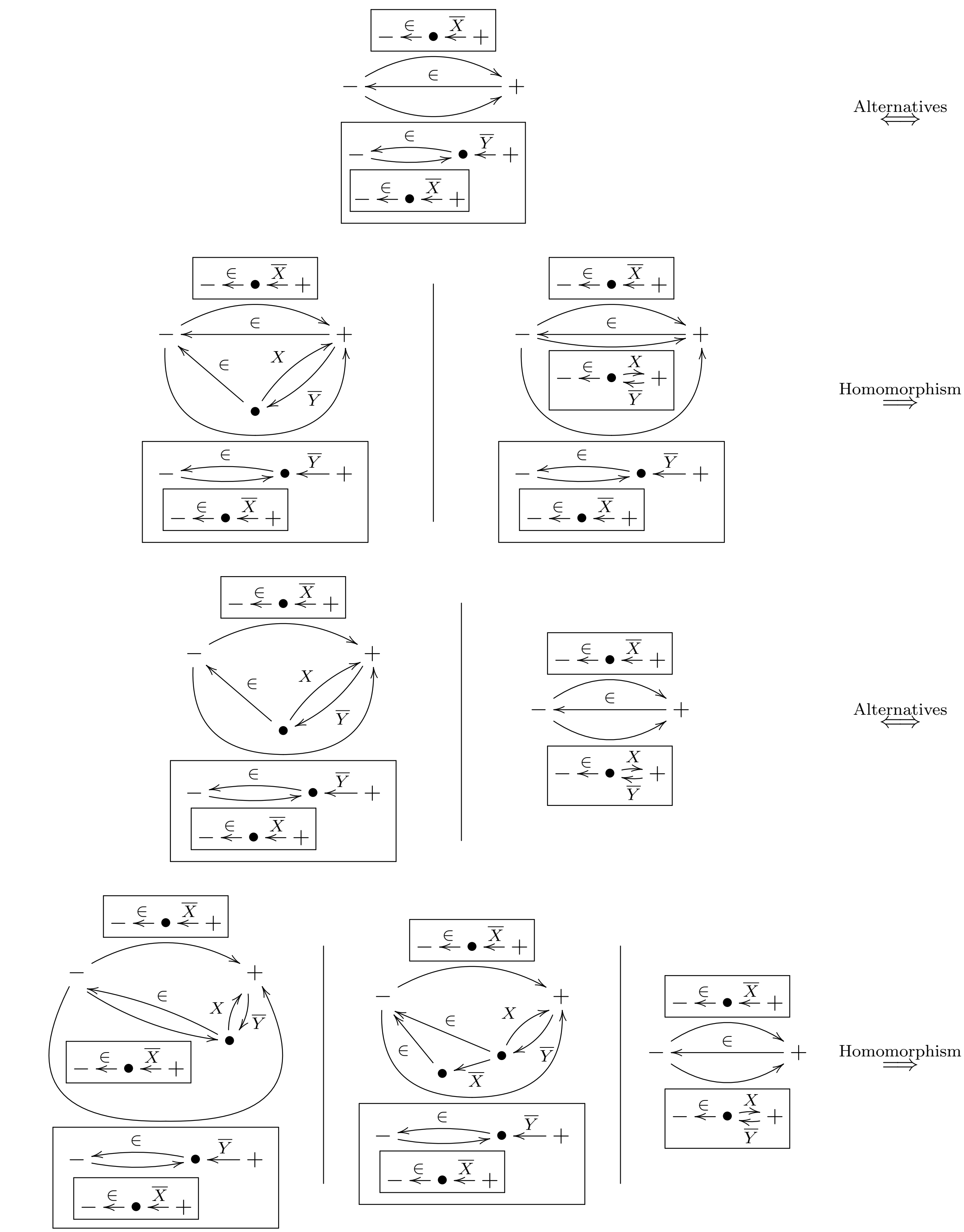

Alternatives

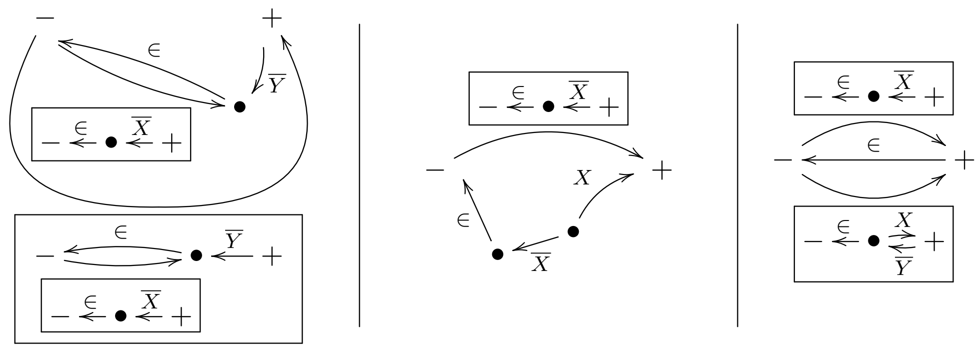

Homomorphism


Inconsistency

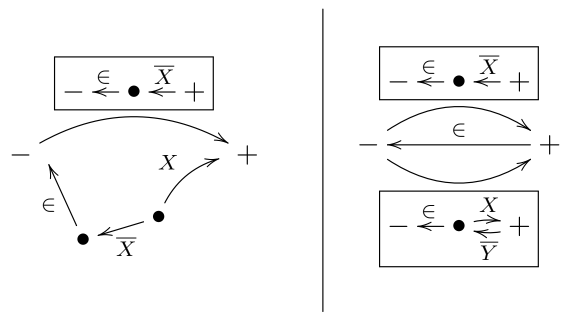


3-5. Diagrammatic proof of $\mathrm{NF}D_{\in^{-1}\cap\overline{\in^{-1}\circ\overline{X}^{-1}}\cap\overline{\left(\in^{-1}\cap\overline{\in^{-1}\circ\overline{X}^{-1}}\right)\circ\overline{Y}^{-1}}} \subseteq \mathrm{NF}D_{\in^{-1}\cap\overline{\in^{-1}\circ\overline{Z}^{-1}}}$:

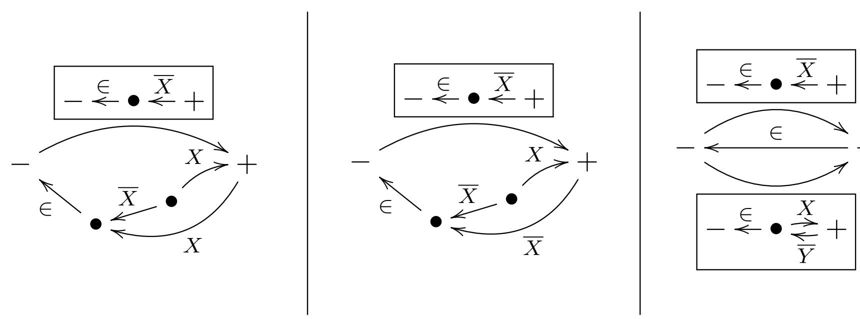




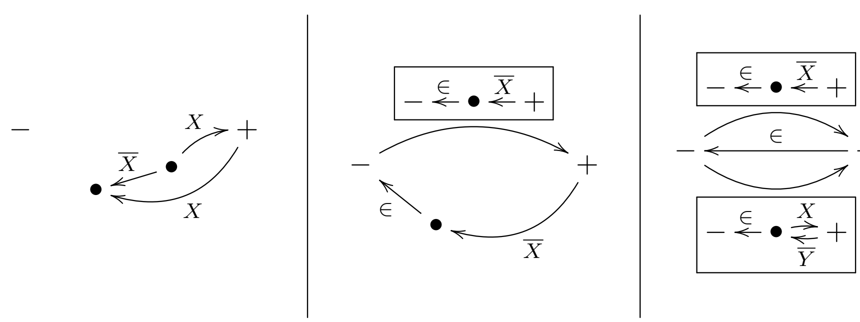
Inconsistency $\Longleftrightarrow$



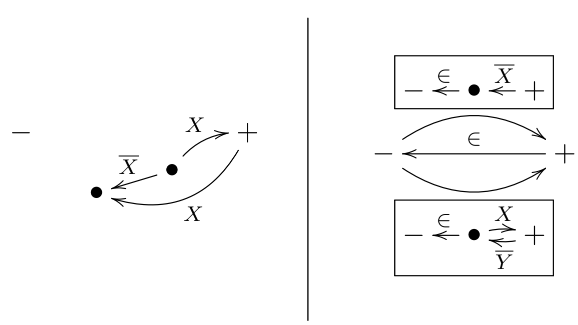
Alternatives $\Longleftrightarrow$



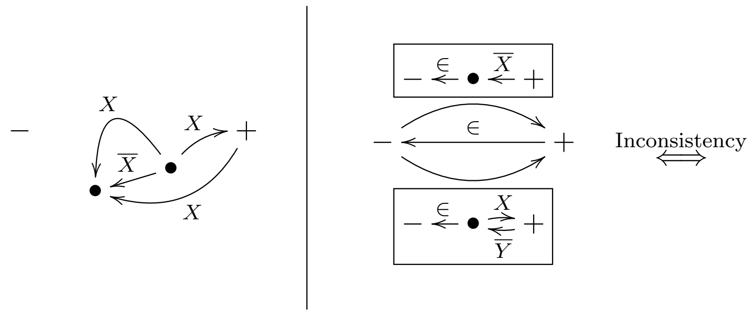
Homomorphism $\Longrightarrow$

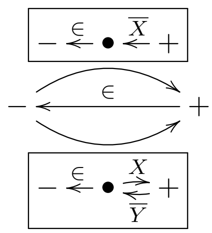


Inconsistency $\Longleftrightarrow$



Hypothesis $\Longrightarrow$





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[5] de Freitas, R.; Veloso, P.; Veloso, S.; Viana, P. A calculus for graphs with complement. *LNAI* 6170:84–98, 2010.